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The Global Existence and Blowing-up Property of Solutions for a Nuclear Model

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In this paper we study the global existence and blowing-up property of solutions for a coupled system of two reaction–diffusion equations under the Neumann conditions, which arises from nuclear reactor dynamics. The estimates of stability regions obtained are the best at the moment. For constant initial data, representations of solutions are given. For a solution blowing-up in finite time an upper bound and a lower bound on the blowing-up time are povided. © 1992 Academic Press, Inc.

1. INTRODUCTION

This paper concerns the parabolic initial-boundary value problem

$$
\begin{aligned}
&u_t - D_1\,\Delta u = u(av - b), && x \in \Omega,\ t > 0,\\
&v_t - D_2\,\Delta v = cu, && x \in \Omega,\ t > 0,\\
&\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, && x \in \partial\Omega,\ t > 0,\\
&u(0, x) = u_0(x), \qquad v(0, x) = v_0(x), && x \in \bar{\Omega},
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$, a, b, c, D_1 and D_2 are given positive constants, and $u_0(x)$ and $v_0(x)$ are nonnegative smooth functions satisfying

$$\frac{\partial u_0}{\partial n} = \frac{\partial v_0}{\partial n} = 0$$

on $\partial\Omega$, which arises from the study of neutron flux and temperature distribution in a nuclear reactor system where the effect of heat conduction

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is taken into consideration; this can be reduced to the special case of $a=c=1$, i.e., to the problem

$$u_t - D_1\,\Delta u = u(v-b), \qquad x\in\Omega,\ t>0, \tag{1}$$

$$v_t - D_2\,\Delta v = u, \qquad x\in\Omega,\ t>0, \tag{2}$$

$$\frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0, \qquad x\in\partial\Omega,\ t>0, \tag{3}$$

$$u(0,x)=u_0(x), \qquad v(0,x)=v_0(x), \qquad x\in\bar{\Omega}, \tag{4}$$

if u and v are replaced by acu and av, respectively. This problem has been discussed by several investigators. In [1], Pao studied the global existence, asymptotic behavior, and blowing-up property of solutions for this problem. His results can be stated as follows.

If $0\leqslant u_0\leqslant (b-v_0)^2/4$, $0\leqslant v_0<b$ then a unique global solution (u,v) of the problem (1)–(4) exists and satisfies

$$\lim_{t\to\infty} u(x,t)=0, \qquad \lim_{t\to\infty} v(x,t)=V_0+U_0,$$

where

$$V_0=\frac{1}{|\Omega|}\int_\Omega v_0(x)\,dx, \qquad U_0=\frac{1}{|\Omega|}\int_0^\infty\int_\Omega u(x,t)\,dx\,dt.$$

On the other hand, if $u_0>b$, $v_0>b$ then the solution (u,v) exists only on $\bar{\Omega}\times[0,T_0)$ for some $T_0<\infty$ and it blows-up to ∞ as $t\to T_0^-$.

According to Pao [1], the stability region of (1)–(4) is given by

$$S=\{(u_0,v_0);\,0\leqslant u_0\leqslant (b-v_0)^2/4,\,0\leqslant v_0<b\},$$

and the instability region is

$$I=\{(u_0,v_0);\,u_0>b,\,v_0>b\}.$$

It is easy to see that there is a big gap between these regions. For instance, if we consider only constant initial data $u_0(x)=u_0$ and $v_0(x)=v_0$ then the stability region S and instability region I for $b<4$ are as shown in Fig. 1.

Recently, Wang [2] improved this result greatly and proved that if one denotes

$$M_1=\max_{\bar{\Omega}} u_0(x), \qquad m_1=\min_{\bar{\Omega}} u_0(x),$$

$$M_2=\max_{\bar{\Omega}} v_0(x), \qquad m_2=\min_{\bar{\Omega}} v_0(x),$$

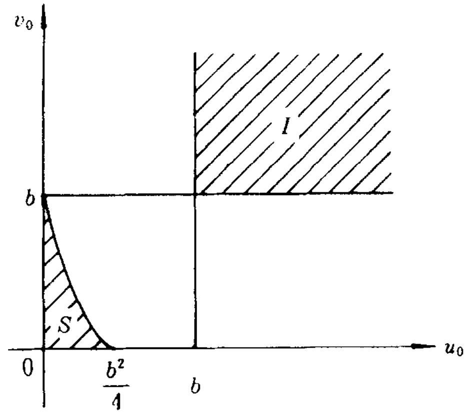

FIGURE 1

then S and I are given by

$$S = \{(u_0, v_0); 0 \leqslant M_1 \leqslant (M_2 - b)^2/2, 0 \leqslant M_2 < b\},$$

$$I = \{(u_0, v_0); u_0 \not\equiv 0, m_2 \geqslant b \text{ or } m_1 \geqslant 3(m_2 - b)^2/2\},$$

which is shown in Fig. 2 for constant u_0 and v_0. It can be seen that there is still a gap between S and I.

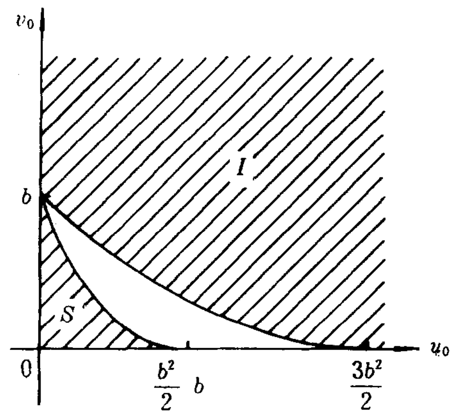

FIGURE 2

In the present paper a representation of the solution of the problem (1)–(4) with constant initial data is obtained, by which the abovementioned gap is filled. For general initial data the global existence and blowing-up property of solutions are also discussed. The asymptotic behavior of global solutions is given when such solutions exist. For a solution blowing-up in finite time an upper bound and a lower bound on the blowing-up time are provided. These bounds are very precise if the oscillations of initial functions are small.

2. Existence and Nonexistence of Global Solutions

It is well known that a solution of the problem (1)–(4) exists (locally at least) which is unique and nonnegative.

Observing first that if $u_0(x) \equiv u_0$ and $v_0(x) \equiv v_0$ are nonnegative constants then solving the problem (1)–(4) can be reduced to solving the following Cauchy problem for ordinary differential equations:

$$p' = p(q-b), \qquad q' = p, \qquad \text{for} \quad t > 0, \tag{5}$$

$$p(0) = u_0, \qquad q(0) = v_0. \tag{6}$$

Theorem 1. *Problem (5)–(6) as well as problem (1)–(4) with nonnegative constant initial data has the global solution $(0, v_0)$ for $u_0 = 0$ and a global solution for $u_0 > 0$ if and only if*

$$u_0 \leqslant (v_0 - b)^2/2 \qquad \textit{and} \qquad v_0 < b.$$

In all cases representations of solutions are given.

Proof. If $u_0 = 0$ then $(0, v_0)$ is clearly a global solution of (5)–(6). So we assume $u_0 > 0$ in the sequel. Set

$$y(t) = p(t) - \tfrac{1}{2}(q(t) - b)^2.$$

It follows from (5)–(6) that $y' = 0$ and

$$y = u_0 - \tfrac{1}{2}(v_0 - b)^2.$$

Thus

$$q' = \tfrac{1}{2}(q-b)^2 + u_0 - \tfrac{1}{2}(v_0 - b)^2. \tag{7}$$

We distinguish three cases corresponding to the sign of $u_0 - \frac{1}{2}(v_0 - b)^2$ to solve (7) and obtain the solution $(p(t), q(t))$ of (5)–(6).

(i) $u_0 - \frac{1}{2}(v_0 - b)^2 = 0$. In this case (5)–(6) has a unique solution $(p(t), q(t))$, which can be represented by

$$p(t) = 4u_0/[2 - (v_0 - b)t]^2 \tag{8}$$

$$q(t) = b + 2(v_0 - b)/[2 - (v_0 - b)t]. \tag{9}$$

It is obvious that when $v_0 < b$ both $p(t)$ and $q(t)$ exist globally, which are nonnegative and satisfy

$$p(t) \downarrow 0, \qquad q(t) \uparrow b \qquad \text{as} \quad t \to \infty.$$

On the other hand, if $v_0 > b$ then $p(t)$ and $q(t)$ exist only on $[0, T)$ with $T = 2/(v_0 - b)$, which are nonnegative on $[0, T)$ and blow-up at $t = T$. Since $u_0 > 0$, $v_0 \neq b$.

(ii) $u_0 - \frac{1}{2}(v_0 - b)^2 < 0$. Now the unique solution $(p(t), q(t))$ of (5)–(6) can be represented by

$$p(t) = \frac{2K\alpha^2 e^{\alpha t}}{(1 - Ke^{\alpha t})^2}, \tag{10}$$

$$q(t) = b - \alpha \frac{Ke^{\alpha t} + 1}{Ke^{\alpha t} - 1}, \tag{11}$$

where

$$\alpha = \sqrt{(v_0 - b)^2 - 2u_0}, \qquad K = \frac{v_0 - b - \alpha}{v_0 - b + \alpha}.$$

It is easily seen that when $v_0 < b$ there must be $K > 1$, and hence $p(t)$ and $q(t)$ defined by (10) and (11) exist globally, which are nonnegative and satisfy

$$p(t) \downarrow 0, \qquad q(t) \uparrow b - \alpha \qquad \text{as} \quad t \to \infty.$$

If $v_0 > b$, and hence $0 < K < 1$, then $p(t)$ and $q(t)$ exist only on $[0, T)$ with $T = (1/\alpha) \ln(1/K)$, which are nonnegative on $[0, T)$ and blow-up at $t = T$.

(iii) $u_0 - \frac{1}{2}(v_0 - b)^2 > 0$. The solution $(p(t), q(t))$ of (5)–(6) now is given by

$$p(t) = \frac{\alpha^2}{2}\left[1 + tg^2\left(\frac{\alpha t}{2} + tg^{-1}\frac{v_0 - b}{\alpha}\right)\right], \tag{12}$$

$$q(t) = b + \alpha tg\left(\frac{\alpha t}{2} + tg^{-1}\frac{v_0 - b}{\alpha}\right), \tag{13}$$

where $\alpha = \sqrt{2u_0 - (v_0 - b)^2}$, which blows-up at

$$t = T := \frac{\pi}{\alpha} + \frac{2}{\alpha} tg^{-1}\frac{b - v_0}{\alpha}$$

no matter what v_0 is.

Thus far the proof of Theorem 1 is complete.

For convenience we denote the solution of (5)–(6) by $(p(t; u_0, v_0), q(t; u_0, v_0))$ and give a comparison lemma which is needed later.

LEMMA. *If* $u_{10} \geqslant u_{20}$ *and* $v_{10} \geqslant v_{20}$ *then*

$$p(t; u_{10}, v_{10}) \geqslant p(t; u_{20}, v_{20}), \qquad q(t; u_{10}, v_{10}) \geqslant q(t; u_{20}, v_{20})$$

for $0 \leqslant t < T$, *where* T *is such a positive number that* $(p(t; u_{10}, v_{10}), q(t; u_{10}, v_{10}))$ *and* $(p(t; u_{20}, v_{20}), q(t; u_{20}, v_{20}))$ *exist on* $[0, T)$, *which can be finite or infinite.*

Proof. If $u_{10} = 0$ then $u_{20} = 0$ and

$$p(t; 0, v_{10}) = 0 = p(t; 0, v_{20}), \qquad q(t; 0, v_{10}) = v_{10} \geqslant v_{20} = q(t; 0, v_{20}).$$

So we assume $u_{10} > 0$ in the rest of the proof.

Set

$$p_1(t) = p(t; u_{10}, v_{10}), \qquad p_2(t) = p(t; u_{20}, v_{20}),$$

$$q_1(t) = q(t; u_{10}, v_{10}), \qquad q_2(t) = q(t; u_{20}, v_{20}).$$

It is easy to see from (5) and the representations (8), (10) and (12) of $p(t)$ that

$$q_1'(t) = p_1(t) > 0, \qquad \forall t \in [0, T). \tag{14}$$

From the equations satisfied by (p_i, q_i) it follows that

$$q_i''(t) = q_i'(t)(q_i(t) - b), \qquad i = 1, 2, 0 \leqslant t < T,$$

which yields for $q(t) = q_1(t) - q_2(t)$

$$q''(t) = (q_2(t) - b)\, q'(t) + q_1'(t)\, q(t), \qquad 0 \leqslant t < T,$$

and hence

$$q'(t) = q'(0) \exp\left\{\int_0^t [q_2(s) - b]\, ds\right\}$$
$$+ \int_0^t q_1'(s)\, q(s) \exp\left\{\int_s^t [q_2(r) - b]\, dr\right\} ds, \qquad 0 \leqslant t < T. \tag{15}$$

Now it is not difficult to deduce the desired conclusion from (14), (15) and

$$q(0) = v_{10} - v_{20} \geqslant 0, \qquad q'(0) = u_{10} - u_{20} \geqslant 0.$$

The next theorem is concerned with the problem (1)–(4) with non-negative initial data which are not necessarily constants.

THEOREM 2. *Let* M_1, M_2, m_1 *and* m_2 *be as defined in Section* 1. *If* $u_0(x) \equiv 0$, *then the problem* (1)–(4) *has the solution* $(0, v(x, t))$, *where* $v(x, t)$ *is the solution of the problem*

$$v_t - D_2 \Delta v = 0, \qquad x \in \Omega,\ t > 0,$$

$$\frac{\partial v}{\partial n} = 0, \qquad x \in \partial\Omega,\ t > 0,$$

$$v(x, 0) = v_0(x), \qquad x \in \bar{\Omega}.$$

If $M_1 \leqslant \frac{1}{2}(M_2 - b)^2$ *and* $M_2 \leqslant b$, *then* (1)–(4) *has a global solution* (u, v) *which satisfies*

$$\lim_{t \to \infty} u(x, t) = 0, \tag{16}$$

$$b - \sqrt{(b - m_2)^2 - 2m_1} \leqslant \lim_{t \to \infty} v(x, t) \leqslant b - \sqrt{(b - M_2)^2 - 2M_1}. \tag{17}$$

While $u_0(x) \not\equiv 0$ *and* $m_2 > b$, *or* $m_1 > \frac{1}{2}(m_2 - b)^2$, *the solution of* (1)–(4) *blows-up in finite time, and the blowing-up time is estimated from above and below by constants depending only on* b, M_1, M_2, m_1 *and* m_2. *The smaller the oscillations of the initial functions, the more precise the estimate on blowing-up time.*

Proof. The conclusion for the case of $u_0(x) \equiv 0$ is trivial. Now we assume $u_0(x) \not\equiv 0$.

It is obvious that the solutions $(p(t; M_1, M_2),\ q(t; M_1, M_2))$ and $(p(t; m_1, m_2),\ q(t; m_1, m_2))$ of (5)–(6) are nonnegative upper and lower solutions of (1)–(4) (for their definitions see [1]), respectively, and

$$p(t; M_1, M_2) \geqslant p(t; m_1, m_2),$$

$$q(t; M_1, M_2) \geqslant q(t; m_1, m_2),$$

by the lemma.

If $M_1 \leqslant \frac{1}{2}(M_2 - b)^2$ and $M_2 \leqslant b$, then $m_1 \leqslant \frac{1}{2}(m_2 - b)^2$ and $m_2 \leqslant b$. Hence according to Theorem 1, $p(t; M_1, M_2)$, $q(t; M_1, M_2)$, $p(t; m_1, m_2)$, and $q(t; m_1, m_2)$ are defined on $t \geqslant 0$. By a known result about the upper and lower solution method (see, e.g., [1]), (1)–(4) has a global solution (u, v) which satisfies

$$p(t; m_1, m_2) \leqslant u(x, t) \leqslant p(t; M_1, M_2),$$

$$q(t; m_1, m_2) \leqslant v(x, t) \leqslant q(t; M_1, M_2).$$

It then follows from the expressions (8)–(11) of p and q that (16) and (17) hold.

By the same argument as above one can see that when $m_2 > b$ the solution of (1)–(4) exists and blows-up in finite time T, which can be estimated according to different cases corresponding to the signs of $M_1 - \frac{1}{2}(M_2 - b)^2$ and $m_1 - \frac{1}{2}(m_2 - b)^2$. For example, if $M_1 < \frac{1}{2}(M_2 - b)^2$ and $m_1 < \frac{1}{2}(m_2 - b)^2$, then by (10) and (11)

$$\frac{1}{\alpha_1} \ln \frac{1}{K_1} \leqslant T \leqslant \frac{1}{\alpha_2} \ln \frac{1}{K_2},$$

where

$$\alpha_1 = \sqrt{(M_2 - b)^2 - 2M_1}, \qquad \alpha_2 = \sqrt{(m_2 - b)^2 - 2m_1},$$

$$K_1 = \frac{M_2 - b - \alpha_1}{M_2 - b + \alpha_1}, \qquad K_2 = \frac{m_2 - b - \alpha_2}{m_2 - b + \alpha_2}.$$

For the case of $m_1 > \frac{1}{2}(m_2 - b)^2$ and $m_2 \leqslant b$, to employ Theorem 1 to get the existence of local solution and the estimate on blowing-up time as before we may distinguish four cases according to

(i) $M_2 > b$, $M_1 = \frac{1}{2}(M_2 - b)^2$,

(ii) $M_2 > b$, $M_1 < \frac{1}{2}(M_2 - b)^2$,

(iii) $M_2 > b$, $M_1 > \frac{1}{2}(M_2 - b)^2$,

(iv) $M_2 \leqslant b$, and hence $M_1 > \frac{1}{2}(M_2 - b)^2$,

and the details are omitted.

3. A Further Discussion on Blowing-up Property

Compared with Theorem 5.3 of [1] and Theorems 1 and 3 of [2], Theorem 2 is an improvement. But it is far from perfect if the initial data are not constants. It is still not clear if the solution exists globally when $m_1 \leqslant \frac{1}{2}(m_2 - b)^2$ and $m_2 \leqslant b < M$, or when $m_1 \leqslant \frac{1}{2}(m_2 - b)^2$, $M_1 > \frac{1}{2}(M_2 - b)^2$ and $M_2 \leqslant b$. In this connection Wang [2] has given an interesting result which says that

If $\bar{v}_0 = (1/|\Omega|) \int_\Omega v_0(x)\, dx \geqslant b$, $u_0(x) \not\equiv 0$, then the solution of (1)–(4) blows-up in finite time.

From this result we know more about the blowing-up property than from Theorem 2. For example, by using this result we can assert that the solution of (1)–(4) blows-up in finite time in the case of $m_1 = 0$, $u_0(x) \not\equiv 0$ and $m_2 = b$, to which Theorem 2 cannot apply.

Now we give this result a supplement.

Denote $\bar{w} = (1/|\Omega|) \int_\Omega w\, dx$ for a function $w(x)$ or $w(x, t)$.

THEOREM 3. *If* $\bar{u}_0 > 0$ *and* $\bar{v}_0 \geqslant b$, *or if* $\bar{u}_0 > (1/2\,|\Omega|)\int_\Omega [v_0(x) - b]^2\,dx$, *then the solution of* (1)–(4) *blows-up in finite time.*

Proof. The proof for the case of $\bar{u}_0 > 0$ and $\bar{v}_0 \geqslant b$ can be found in [2]. We now assume

$$\bar{u}_0 > \frac{1}{2\,|\Omega|} \int_\Omega [v_0(x) - b]^2\,dx.$$

Let (u, v) be the solution of (1)–(4). Integrating (1) and (2) over Ω, dividing the thus obtained equalities by $|\Omega|$, and taking (3) into account, we get

$$\bar{u}' = \frac{1}{|\Omega|} \int_\Omega u(v - b)\,dx \tag{18}$$

and

$$\bar{v}' = \bar{u}, \tag{19}$$

where $'$ stands for d/dt. Applying (2) again to (18) and taking (4) into account we obtain

$$\begin{aligned} \bar{u} &= \bar{u}_0 + \frac{1}{|\Omega|} \int_0^t ds \int_\Omega (v_t - D_2\,\Delta v)(v - b)\,dx \\ &= \bar{u}_0 - \frac{1}{2\,|\Omega|} \int_\Omega (v_0 - b)^2\,dx + \frac{1}{2\,|\Omega|} \int_\Omega (v - b)^2\,dx \\ &\pm \frac{D_2}{|\Omega|} \int_0^t ds \int_\Omega |\nabla v|^2\,dx. \end{aligned} \tag{20}$$

Set

$$A = \bar{u}_0 - \frac{1}{2\,|\Omega|} \int_\Omega (v_0 - b)^2\,dx.$$

By the assumption $A > 0$. From (19) and (20) it follows that

$$\bar{v}' \geqslant A + \frac{1}{2\,|\Omega|} \int_\Omega (v - b)^2\,dx \geqslant A + \frac{1}{2}(\bar{v} - b)^2,$$

which implies

$$\bar{v}(t) \geqslant b + \alpha tg\left(\frac{\alpha t}{2} + tg^{-1}\frac{\bar{v}_0 - b}{\alpha}\right),$$

where

$$\alpha=\left[2\bar{u}_0-\frac{1}{|\Omega|}\int_\Omega (v_0-b)^2\,dx\right]^{1/2}.$$

This shows that $\bar{v}(t)$ blows-up in finite time.

References

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